Solution 3 by Bruno Salgueiro Fanego, Viveiro, Spain

The required number of digits is 29 because, as we shall show, $10^{28} \le 2^{96} < 10^{29}$. More exactly, we shall prove that $1 < \dfrac{2^{96}}{10^{28}} < 10$. Since

$$\frac{2^{96}}{10^{28}} = \left(\frac{2^{24}}{10^7}\right)^4 = \left(\frac{\left(2^{12}\right)^2}{10^7}\right)^4 = \left(\frac{4096^2}{10^7}\right)^4 = \left(\frac{1,6777216 \cdot 10^7}{10^7}\right)^4 = (1,6777216)^4,$$

we obtain that

$1^4 < \dfrac{2^{96}}{10^{28}} < 1,68)^4$, that is $1 < \dfrac{2^{96}}{10^{28}} < (2.8224)^2$ and, hence, $1 < \dfrac{2^{96}}{10^{28}} < 3^2 < 10$.

Note: another way to show that $10^{28} < 2^{96}$ is, for example:

$$\left.\begin{array}{r}5^2 < 2^5 \\ 5 < 2^3\end{array}\right\} \Rightarrow \left.\begin{array}{r}5^2 < 2^5 \\ 5^3 < 2^9\end{array}\right\} \Rightarrow 5^5 < 2^5 \cdot 5^3 < 2^{12} \Rightarrow \left.\begin{array}{r}5^5 < 2^{12} \\ 5^2 < 2^5\end{array}\right\} \quad \Rightarrow 5^7 < 2^5 \cdot 5^5 < 2^{17} \Rightarrow$$

$$\Rightarrow 2^7 \cdot 5^7 < 2^{24} \Rightarrow$$

$$\Rightarrow \left(10^7\right)^4 < \left(2^{24}\right)^4 \Rightarrow$$

$$\Rightarrow 10^{28} < 2^{96}.$$

Solution 4 by Toshihiro Shimizu, Kawasaki, Japan

Since $10^3 < 2^{10} = 1024 < 1.03 \times 10^3$ and $2^{96} = \left(2^{10}\right)^9 \times 2^6 = \left(2^{10}\right)^9 \times 10 \times 6.4$ we have

$$6.4 \times 10 \times 10^{3\times9} < 2^{96} < 6.4 \times 10 \times 10^{3\times9} \times (1.03)^9.$$

We evaluate 1.03^9. We have $1.03 \times 1.03 \times 1.03 = 1.0609 \times 1.03 = 1.092727 < 1.1$ and $1.1 \times 1.1 \times 1.1 = 1.331 < 1.4$ (I never use calculator.) Therefore, we have

$$10^{28} < 6.4 \times 10^{28} < 2^{96} < 6.4 \times 1.4 \times 10^{28} = 8.96 \times 10^{28} < 10^{29}.$$

Therefore, the number of digits in 2^{96} is 29.

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Kee-Wai Lau, Hong Kong, China; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposers.

● **5435:** *Proposed by Valcho Milchev, Petko Rachov Slaveikov Seconday School, Bulgaria*

Find all positive integers a and b for which $\dfrac{a^4 + 3a^2 + 1}{ab - 1}$ is a positive integer.

Solution 1 by Moti Levy, Rehovot, Israel

This solution is based on similar problem and solution which appeared in [1].

$\dfrac{a^4 + 3a^2 + 1}{ab - 1}$ may be replaced by equivalent expression with *symmetric* polynomial in the numerator.

Indeed,
$$\frac{a^4 + 3a^2 + 1}{ab - 1} = \frac{a^2\left(a^2 + b^2 + 3\right) - (ab - 1)(ab + 1)}{ab - 1}.$$

Now, a and $ab - 1$ satisfy the equation $b * a + (-1) * (ab - 1) = 1$, which implies that a and $ab - 1$ are relatively prime and clearly a^2 and $ab - 1$ are also relatively prime. Thus, $\dfrac{a^4 + 3a^2 + 1}{ab - 1}$ is a positive integer if and only if $\dfrac{a^2 + b^2 + 3}{ab - 1}$ is a positive integer.

We call the ordered pair (a, b) a *solution* if

$$\frac{a^2 + b^2 + 3}{ab - 1} = m, \tag{1}$$

where m is a positive integer. The set of solutions is not empty since $(1, 2)$ is a solution. We exclude (a, a) from the set of solutions since $\dfrac{2a^2 + 3}{a^2 - 1} = 2 + \dfrac{5}{a^2 - 1} \notin N$ for all $a > 0$. Equation (1) is re-written as follows

$$a^2 - mab + b^2 = -(m + 3). \tag{2}$$

It is easily verified (see (3)) that if (a, b) is a solution then $(ma - b, a)$ is a solution as well.
$$(ma - b)^2 - m(ma - b)a + a^2 = a^2 - mab + b^2, \tag{3}$$

Let (a_0, b_0) be the "smallest" solution in the sense that $a_0 + b_0 \leq a + b$, where (a, b) is any solution.

$$a_0 + b_0 \leq (ma_0 - b_0) + a_0,$$

or
$$\frac{2b_0}{a_0} \leq m. \tag{4}$$

$$\frac{2b_0}{a_0} \leq \frac{a_0^2 + b_0^2 + 3}{a_0 b_0 - 1}$$
$$0 \leq -2a_0 b_0^2 + 2b_0 + a_0^3 + 3a_0 \tag{5}$$

Let $(a_0, a_0 + k)$ be a solution. Then substituting in (5) gives,
$$0 \leq -2a_0(a_0 + k)^2 + 2(a_0 + k) + a_0^3 + 3a_0$$
$$= -2k^2 a_0 - 4ka_0^2 + 2k - a_0^3 + 5a_0.$$

Solving $-2k^2 a_0 - 4ka_0^2 + 2k - a_0^3 + 5a_0 \geq 0$, we get

$$\frac{1}{2a_0}\left(1 - 2a_0^2 - \sqrt{6a_0^2 + 2a_0^4 + 1}\right) \leq k \leq \frac{1}{2a_0}\left(1 - 2a_0^2 + \sqrt{6a_0^2 + 2a_0^4 + 1}\right),$$

hence, k will have positive values only if

$$\sqrt{6a_0^2 + 2a_0^4 + 1} + 1 \geq 2a_0^2.$$

This inequality holds for $a_0 = 1$ and $a_0 = 2$. For $a_0 = 1$, possible values for k are 1 or 2; for $a_0 = 2$, possible value for k is 1.

Thus we have to check the following set of potential solutions: $\{(1,2),(1,3),(2,1)\}$. Clearly $(1,2)$ and $(2,1)$ are solutions, but $(1,3)$ is not.

For $(1,2)$ and $(2,1)$ the value of m is 8. We conclude that the sole value of m is 8.

It follows from (3) that the pairs (a_n, b_n) (and by symmetry (b_n, a_n)), which satisfy condition (1) are expressed by the recurrence formulas

$$a_{n+1} = 8a_n - b_n,$$
$$b_{n+1} = a_n,$$

which are equivalent to the recurrence formulas

$$a_{n+2} = 8a_{n+1} - a_n, \tag{6}$$
$$b_{n+2} = 8b_{n+1} - b_n.$$

We have two sets of initial conditions:

1) $a_0 = 1$, $a_1 = 6, b_0 = 2, b_1 = 1$; the pairs resulting from these initial conditions are $(1,2)$, $(6,1)$, $(47,6)$, $(370,47)$,....

$$a_n = \left(\frac{1}{2} - \frac{1}{\sqrt{15}}\right)\left(4 - \sqrt{15}\right)^n + \left(\frac{1}{2} + \frac{1}{\sqrt{15}}\right)\left(4 + \sqrt{15}\right)^n,$$
$$b_n = \left(1 + \frac{7}{2\sqrt{15}}\right)\left(4 - \sqrt{15}\right)^n + \left(1 - \frac{7}{2\sqrt{15}}\right)\left(4 + \sqrt{15}\right)^n.$$

2) $a_0 = 2$, $a_1 = 15, b_0 = 1, b_1 = 2$; the pairs resulting from these initial conditions are $(2,1)$, $(15,2)$, $(118,15)$, $(929,118)$, ...

$$a_n = \left(1 - \frac{7}{2\sqrt{15}}\right)\left(4 - \sqrt{15}\right)^n + \left(1 + \frac{7}{2\sqrt{15}}\right)\left(4 + \sqrt{15}\right)^n,$$
$$b_n = \left(\frac{1}{2} + \frac{1}{\sqrt{15}}\right)\left(4 - \sqrt{15}\right)^n + \left(\frac{1}{2} - \frac{1}{\sqrt{15}}\right)\left(4 + \sqrt{15}\right)^n.$$

Reference:

[1] La Gaceta de la RSME, Vol. 18 (2015), No. 1, "*Solution to Problem 241, by Roberto de la Cruz Moreno*".

Solution 2 by Anthony Bevelacqua, University of North Dakota, Grand Forks, ND

1) There are no solutions to our problem with $a = b$. We have $a^4 + 3a^2 + 1 \equiv 5 \bmod (a^2 - 1)$. Assume there is a solution with $a = b$. Then $a^2 - 1$ divides $a^4 + 3a^2 + 1$ so $a^4 + 3a^2 + 1 \equiv 0 \bmod (a^2 - 1)$. Thus $5 \equiv 0 \bmod (a^2 - 1)$ and so $a^2 - 1$ divides 5. But then $a^2 = 2$ or $a^2 = 6$, a contradiction in either case.

2) The only solutions with $a \le 4$ are $(a, b) = (1,2)$, $(2,1)$, $(1,6)$ and $(2,15)$. Suppose (a, b) is a solution to our problem. If $a = 1$ then $b - 1$ divides 5 so $b - 1 = 1$ or $b - 1 = 5$. Both $(1,2)$ and $(1,6)$ are solutions. If $a = 2$ then $2b - 1$ divides 29 so $2b - 1 = 1$ or $2b - 1 = 29$. Both $(2,1)$ and $(2,15)$ are solutions. If $a = 3$ then $3b - 1$ divides 109 so $3b - 1 = 1$ or $3b - 1 = 109$, a contradiction. If $a = 4$ then $4b - 1$ divides $305 = 5 \cdot 61$ so $4b - 1 \in \{1, 5, 61, 305\}$, a contradiction.

3) $ab - 1$ divides $a^4 + 3a^2 + 1$ if and only if $ab - 1$ divides $a^2 + b^2 + 3$.
We have

$$
\begin{aligned}
(ab - 1)(a^3b + 3ab + a^2 + 3) &= a^4b^2 + 3a^2b^2 + a^3b + 3ab - a^3b - 3ab - a^2 - 3 \\
&= a^4b^2 + 3a^2b^2 - a^2 - 3
\end{aligned}
$$

and so
$$
b^2(a^4 + 3a^2 + 1) - (ab - 1)(a^3b + 3ab + a^2 + 3) = a^2 + b^2 + 3.
$$

Thus if $ab - 1$ divides $a^4 + 3a^2 + 1$ then $ab - 1$ divides $a^2 + b^2 + 3$. Conversely suppose $ab - 1$ divides $a^2 + b^2 + 3$. Then $ab - 1$ divides $b^2(a^4 + 3a^2 + 1)$. Since $ab - 1$ and b^2 are relatively prime we have that $ab - 1$ divides $a^4 + 3a^2 + 1$.

Now if $k > 0$ and (a, b) is a solution to $a^2 + b^2 + 3 = k(ab - 1)$ then b is a root of the polynomial $a^2 + x^2 + 3 = k(ax - 1)$ which can be rewritten as
$x^2 - kax + (a^2 + 3 + k) = 0$. Thus if b' is the other root we have, by Vieta's formulas, $b + b' = ka$ and $bb' = a^2 + 3 + k$. The first shows that b' is an integer and the second shows that $b' > 0$. Thus (a, b') is another solution to $a^2 + b^2 + 3 = k(ab - 1)$.

4) If $ab - 1$ divides $a^2 + b^2 + 3$ then $a^2 + b^2 + 3 = 8(ab - 1)$. Suppose there are positive integers a, b, k such that $a^2 + b^2 + 3 = k(ab - 1)$. For this fixed k let S be the set of all positive integer pairs (a, b) such that $a^2 + b^2 + 3 = k(ab - 1)$. Choose an $(a, b) \in S$ such that $a + b$ is minimal. Without loss of generality we have $a \le b$. Since $a \ne b$ by 1) we have $a < b$. Now (a, b') is another solution. Since $a + b$ is minimal we have $a + b \le a + b'$ and hence $b \le b'$. Thus

$$
b^2 \le bb' = a^2 + 3 + k \implies k \ge b^2 - a^2 - 3
$$

and so

$$
\begin{aligned}
a^2 + b^2 + 3 &= k(ab - 1) \\
&\ge (b^2 - a^2 - 3)(ab - 1) \\
&= ab^3 - b^2 - a^3b + a^2 - 3ab + 3.
\end{aligned}
$$

Hence

$$
3ab + 2b^2 \ge ab^3 - a^3b \implies 3a + 2b \ge ab^2 - a^3.
$$

Since $a < b$ we have $3a + 2b < 5b$ and $ab^2 - a^3 = a(b + a)(b - a) > ab$. Thus $5b > ab$ and so $a < 5$. By 2) the only possible (a, b) are then $(1, 2)$, $(1, 6)$, and $(2, 15)$. Each of these gives $k = 8$.

Thus 3) and 4) show that our original problem is equivalent to finding all positive integers a and b such that $a^2 + b^2 + 3 = 8(ab - 1)$. We could rewrite this as $(a - 4b)^2 - 15b^2 = -11$ and apply the theory of equations of the form $x^2 - Dy^2 = N$ as found in, say, section 58 of Nagell's *Number Theory*. Instead we will determine the solutions by "Vieta jumping" as in the proof of (4).

Let S be the set of all positive integers pairs (a, b) such that $a^2 + b^2 + 3 = 8(ab - 1)$. Clearly if $(a, b) \in S$ then $(b, a) \in S$, and, by 1) there are no $(a, b) \in S$ with $a = b$. Recall that if $(a, b) \in S$ then $(a, b') \in S$ where $b + b' = 8a$ and $bb' = a^2 + 11$.

5) For any $(a, b) \in S$ define $\rho(a, b) = (b', a)$ and $\lambda(a, b) = (b, 8b - a)$. Then $\rho(a, b) \in S$, $\lambda(a, b) \in S$, and $\lambda(\rho(a, b)) = (a, b)$.

Let $(a, b) \in S$. We have $(a, b') \in S$ and hence $\rho(a, b) = (b', a) \in S$. Now

$$
\begin{aligned}
b^2 + (8b - a)^2 + 3 &= 64b^2 - 16ab + (a^2 + b^2 + 3) \\
&= 64b^2 - 16ab + 8(ab - 1) \\
&= 64b^2 - 8ab - 8 \\
&= 8(b(8b - a) - 1)
\end{aligned}
$$

so $\lambda(a, b) = (b, 8b - a) \in S$. Finally,

$$
\lambda(\rho(a, b)) = \lambda(b', a) = (a, 8a - b')
$$

where

$$
8a - b' = 8a - \frac{a^2 + 11}{b} = \frac{8ab - a^2 - 11}{b} = \frac{b^2}{b} = b.
$$

6) The only $(a, b) \in S$ such that $a < b \le 10$ are $(a, b) = (1, 2)$ and $(1, 6)$.

Since $a^2 + b^2 + 3 \equiv 0 \bmod 8$ we see that a and b must have opposite parity and neither can be divisible by 4. Moreover the only such solutions with a or b less than 4 are $(1, 2)$ and $(1, 6)$ by 2). This leaves only

$$
(a, b) = (5, 6), (6, 7), (6, 9), (5, 10), (7, 10), (9, 10)
$$

and none of these satisfy $a^2 + b^2 + 3 = 8(ab - 1)$.

7) Let $(a, b) \in S$ such that $b \ge 11$. If $a < b$ then $b' < a$

Suppose first that $b' \le 10$. Assume $a \le b'$. Since $(a, b') \in S$ we have $a \ne b'$. Thus $a < b' \le 10$. So, by 6), we must have $a = 1$. But if $a = 1$ we have $b = 1$ or $b = 6$, a contradiction with $b \ge 11$. Hence $b' < a$.

Suppose now that $b' \ge 11$. Again assume $a \le b'$. Then, as in the last paragraph, $a < b'$. We have

$$
bb' = a^2 + 11 < (b')^2 + 11 \implies b < b' + \frac{11}{b'} \le b' + 1
$$

and so $b \le b'$. Now swapping b and b' we have

$$
bb' = a^2 + 11 < b^2 + 11 \implies b' < b + \frac{11}{b} \le b + 1
$$

and so $b' \le b$. Thus $b = b'$. Since $8a = b + b' = 2b$ we have $b = 4a$. But then

$$
a^2 + 16a^2 + 3 = 8(4a^2 - 1) \implies 11 = 15a^2,
$$

a contradiction. Hence $b' < a$.

Finally,

8) $(a, b) \in S$ if and only if $\{a, b\} = \{s_n, s_{n+1}\}$ or $\{a, b\} = \{t_n, t_{n+1}\}$ for $n \ge 0$ where

$$
s_0 = 1, \ s_1 = 2, \ \text{and } s_n = 8s_{n-1} - s_{n-2} \text{ for } n \ge 2
$$

and

$$
t_0 = 1, \ t_1 = 6, \ \text{and } t_n = 8t_{n-1} - t_{n-2} \text{ for } n \ge 2.
$$

Note that $\lambda^n(1,2) = (s_n, s_{n+1})$ and $\lambda^n(1,6) = (t_n, t_{n+1})$ for all $n \geq 0$.

Since $(1,2)$ and $(1,6) \in S$ we see that $(a,b) \in S$ for any $\{a,b\} = \{s_n, s_{n+1}\}$ or $\{a,b\} = \{t_n, t_{n+1}\}$ and $n \geq 0$ by (5).

Now suppose $(a,b) \in S$. Since $(b,a) \in S$ as well, we can suppose without loss of generality that $a < b$. By 5) and 7) there exists an integer $d \geq 0$ such that $\rho^d(a,b) = (a^*, b^*)$ with $a^* < b^* \leq 10$. By (6) we must have $\rho^d(a,b) = (1,2)$ or $\rho^d(a,b) = (1,6)$. Since $(a,b) = \lambda^d(\rho^d(a,b))$ we have $(a,b) = \lambda^d(1,2)$ or $(a,b) = \lambda^d(1,6)$.

Thus $ab - 1$ divides $a^4 + 3a^2 + 1$ if and only if a and b are consecutive elements of either of the sequences s_n or t_n given above. Since the first few terms of s_n are $1, 2, 15, 118, 929, 7314, 57583, \ldots$ and the first few terms of t_n are $1, 6, 47, 370, 2913, 22934, 180559, \ldots$ the first few solutions to our problem (with $a \leq b$) are
$$(a,b) = (1,2), (2,15), (15,118), (118,929), (929,7314), (7314,57583), \ldots$$
and
$$(a,b) = (1,6), (6,47), (47,370), (370,2913), (2913,22934), (22934,180559), \ldots$$

Also solved by Ed Gray, Highland Beach, FL; Kenneth Korbin, NewYork, NY; Toshihiro Shimizu, Kawasaki, Japan; Anna V. Tomova (three solutions), Varna, Bulgaria, and the proposer.

- **5436:** *Proposed by Arkady Alt, San Jose, CA*

Find all values of the parameter t for which the system of inequalities
$$\mathbf{A} = \begin{cases} \sqrt[4]{x+t} \geq 2y \\ \sqrt[4]{y+t} \geq 2z \\ \sqrt[4]{z+t} \geq 2x \end{cases}$$

a) has solutions;

b) has a unique solution.

Solution by the Proposer

a) Note that $(\mathbf{A}) \Longleftrightarrow \begin{cases} t \geq 16y^4 - x \\ t \geq 16z^4 - y \\ t \geq 16x^4 - z \end{cases} \Longrightarrow 3t \geq 16y^4 - x + 16z^4 - y + 16x^4 - z =$

$\left(16x^4 - x\right) + \left(16y^4 - y\right) + \left(16z^4 - z\right) \geq 3 \min_x \left(16x^4 - x\right) \Longrightarrow t \geq \min_x \left(16x^4 - x\right).$

For $x \in \left(0, \dfrac{1}{16}\right)$, using the AM-GM Inequality, we obtain

$x - 16x^4 = x\left(1 - 16x^3\right) = \sqrt[3]{x^3\left(1 - 16x^3\right)^3} = \sqrt[3]{\dfrac{\left(48x^3\right)\left(1 - 16x^3\right)^3}{48}} \leq$

$\sqrt[3]{\dfrac{1}{48} \cdot \left(\dfrac{48x^3 + 3 - 3 \cdot 16x^3}{4}\right)^4} = \sqrt[3]{\dfrac{1}{48} \cdot \left(\dfrac{3}{4}\right)^4} = \dfrac{3}{16}$. And since $x - 16x^4 \leq 0$ for

$x \notin \left(0, \dfrac{1}{16}\right)$, then for all x the inequality $x - 16x^4 \leq \dfrac{3}{16}$ holds. Since the upper bound is $\dfrac{3}{16}$ for values

$x - 16x^4$ is attainable when $x = \dfrac{1}{4}$, then $\max\left(x - 16x^4\right) = \dfrac{3}{16} \iff$

$\min\limits_{x}\left(16x^4 - x\right) = -\dfrac{3}{16}$.

Thus $t \geq -\dfrac{3}{16}$ is a necessary condition for the solvability of system **(A).**

Let's prove sufficiency.

Let $t \geq -\dfrac{3}{16}$. Since function $h(x)$ is continuous in R and $\min\limits_{x}\left(16x^4 - x\right) = -\dfrac{3}{16}$, then

$\left[-\dfrac{3}{16}, \infty\right)$ is the range of $h(x)$. This means that for any $t \geq -\dfrac{3}{16}$ the equation $16x^4 - x = t$

has solution in R and since for any u which is a solution of the equation $16x^4 - x = t$

the triple $(x, y, z) = (u, u, u,)$ is a solution of the system **(A)** then for such t system **(A)**

solvable as well.

Remark.

Actually the latest reasoning about the solvability of system **(A)** if $t \geq -\dfrac{3}{16}$ is redundant

for **(a)** because suffices to note that for such t the triple $(x, y, z) = \left(\dfrac{1}{4}, \dfrac{1}{4}, \dfrac{1}{4}\right)$ satisfies to

(A).

But for **(b)** criteria of solvability of equation $16x^4 - x = t$ in form of inequality

$t \geq -\dfrac{3}{16}$ is

important.

b) Note that system **(A)** always have more the one solution if $t > -\dfrac{3}{16}$.

Indeed, let for any $t_1, t_2 \in \left(-\dfrac{3}{16}, t\right)$ such that $t_1 \neq t_2$ equation $16u^4 - u = t_i$ has

solution $u_i, i = 1, 2$.

Then $u_1 \neq u_2$ and two distinct triples $(u_1, u_1, u_1), (u_2, u_2, u_2)$ satisfy to the system **(A).**

Let $t = -\dfrac{3}{16}$. Then $-\dfrac{3}{16} \geq 16y^4 - x \implies -\dfrac{3}{16} + x - y \geq 16y^4 - y \geq -\dfrac{3}{16}$.

Hereof $x - y \geq 0 \iff x \geq y$. Similarly $-\dfrac{3}{16} \geq 16z^4 - y$ and $-\dfrac{3}{16} \geq 16x^4 - z$ implies

$y \geq z$ and $z \geq x$, respectively. Thus in that case $x = y = z$ and all solutions of the

system **(A)** are represented by solutions of one equation $16x^4 - x = -\dfrac{3}{16} \iff$

$16x^4 - x + \dfrac{3}{16} = 0 \iff 256x^4 - 16x + 3 = 0$ which has only root $\dfrac{1}{4}$ because

$256x^4 - 16x + 3 = (4x - 1)^2\left(16x^2 + 8x + 3\right)$.

Thus, system **(A)** has unique solution iff $t = \dfrac{1}{4}$.

Also solved by Ed Gray, Highland Beach,FL; Kee-Wai Lau, Hong Kong, China; Moti Levy, Rehovot, Israel; David Stone and John Hawkins, Georgia

Southern University, Statesboro, GA, and Toshihiro Shimizu, Kawasaki, Japan.

- **5437:** *Proposed by José Luis Díaz-Barrero, Barcelona Tech, Barcelona, Spain*

Let $f : C - \{2\} \to C$ be the function defined by $f(z) = \dfrac{2-3z}{z-2}$. If $f^n(z) = (\underbrace{f \circ f \circ \ldots \circ f}_{n})(z)$, then compute $f^n(z)$ and $\lim\limits_{n \to +\infty} f^n(z)$.

Solution 1 by Dionne Bailey, Elsie Campbell, and Charles Diminnie, Angelo State University, San Angelo, TX

Assume first that $z \neq 2$ and $f^n(z)$ exists for all $n \geq 1$. Then, direct computation yields

$$f^2(z) = \frac{10 - 11z}{5z - 6} \quad \text{and} \quad f^3(z) = \frac{42 - 43z}{21z - 22}. \tag{1}$$

When these are combined with the formula for $f(z)$, it appears that there is a sequence $\{x_n\}$ of positive integers such that

$$f^n(z) = \frac{2x_n - (2x_n + 1)z}{x_n z - (x_n + 1)} \tag{2}$$

for all $n \geq 1$. Since $f(z) = \dfrac{2-3z}{z-2}$, we have $x_1 = 1$. Further, if (2) holds for some $n \geq 1$, then

$$f^{n+1}(z) = f(f^n(z))$$

$$= \frac{2 - 3f^n(z)}{f^n(z) - 2}$$

$$= \frac{2 - 3\left[\dfrac{2x_n - (2x_n + 1)z}{x_n z - (x_n + 1)}\right]}{\left[\dfrac{2x_n - (2x_n + 1)z}{x_n z - (x_n + 1)}\right] - 2}$$

$$= \frac{2[x_n z - (x_n + 1)] - 3[2x_n - (2x_n + 1)z]}{[2x_n - (2x_n + 1)z] - 2[x_n z - (x_n + 1)]}$$

$$= \frac{(8x_n + 2) - (8x_n + 3)z}{(4x_n + 1)z - (4x_n + 2)}.$$

This suggests that $x_{n+1} = 4x_n + 1$ for $n \geq 1$. These conditions on $\{x_n\}$ are consistent with the formula for $f(z)$ and property (2). Note finally that

$$x_1 = 1 = \frac{3}{3} = \frac{4-1}{3}, \quad x_2 = 5 = \frac{15}{3} = \frac{4^2 - 1}{3}, \quad \text{and} \quad x_3 = 21 = \frac{63}{3} = \frac{4^3 - 1}{3}.$$

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This leads us to conjecture that $x_n = \dfrac{4^n - 1}{3}$ and hence,

$$f^n(z) = \frac{2\left(\dfrac{4^n-1}{3}\right) - \left[2\left(\dfrac{4^n-1}{3}\right)+1\right]z}{\left(\dfrac{4^n-1}{3}\right)z - \left[\left(\dfrac{4^n-1}{3}\right)+1\right]} = \frac{2(4^n-1)-(2\cdot 4^n+1)z}{(4^n-1)z-(4^n+2)}$$

for all $n \geq 1$.

If $f^n(z)$ exists for all $n \geq 1$, let $P(n)$ be the statement

$$f^n(z) = \frac{2(4^n-1)-(2\cdot 4^n+1)z}{(4^n-1)z-(4^n+2)}. \tag{3}$$

If $n = 1$,

$$\frac{2(4-1)-(2\cdot 4+1)z}{(4-1)z-(4+2)} = \frac{6-9z}{3z-6}$$

$$= \frac{2-3z}{z-2}$$

and thus, $P(1)$ is true. Assume that $P(n)$ is true, i.e.,

$$f^n(z) = \frac{2(4^n-1)-(2\cdot 4^n+1)z}{(4^n-1)z-(4^n+2)}$$

for some $n \geq 1$. Then,

$$f^{n+1}(z) = f(f^n(z))$$

$$= \frac{2-3\left[\dfrac{2(4^n-1)-(2\cdot 4^n+1)z}{(4^n-1)z-(4^n+2)}\right]}{\left[\dfrac{2(4^n-1)-(2\cdot 4^n+1)z}{(4^n-1)z-(4^n+2)}\right]-2}$$

$$= \frac{2[(4^n-1)z-(4^n+2)]-3[2(4^n-1)-(2\cdot 4^n+1)z]}{[2(4^n-1)-(2\cdot 4^n+1)z]-2[(4^n-1)z-(4^n+2)]}$$

$$= \frac{[2(4^n-1)+3(2\cdot 4^n+1)]z-[2(4^n+2)+6(4^n-1)]}{[2(4^n-1)+2(4^n+2)]-[2\cdot 4^n+1+2(4^n-1)]z}$$

$$= \frac{(2\cdot 4^{n+1}+1)z-2(4^{n+1}-1)}{(4^{n+1}+2)-(4^{n+1}-1)z}$$

$$= \frac{2(4^{n+1}-1)-(2\cdot 4^{n+1}+1)z}{(4^{n+1}-1)z-(4^{n+1}+2)}$$

and therefore, $P(n+1)$ is also true. By Mathematical Induction, $P(n)$ is true for all $n \geq 1$.

Because formula (3) required the assumption that $f^n(z)$ exists for all $n \geq 1$, we need to determine if there are points $z \in C \setminus \{2\}$ for which there is a positive integer m such that

$f^n(z)$ does not exist for $n > m$. The existence of $f^n(z)$ requires that z, $f(z)$, ...,
$f^{n-1}(z) \neq 2$. Therefore, we have to find all points z for which $f^m(z) = 2$ for some
$m \geq 1$. One way to do this is to consider the inverse function

$$f^{-1}(z) = \frac{2z+2}{z+3}$$

and describe

$$f^{-m}(z) = \left(\underbrace{f^{-1} \circ f^{-1} \circ \ldots \circ f^{-1}}_{m}\right)(z)$$

in a manner similar to that used to find formula (3). If we do so, we see that for $z \neq -3$,

$$f^{-m}(z) = \frac{(4^m + 2)z + 2(4^m - 1)}{(4^m - 1)z + 2 \cdot 4^m + 1}.$$

In particular,

$$f^{-m}(2) = \frac{(4^m + 2) \cdot 2 + 2(4^m - 1)}{(4^m - 1) \cdot 2 + 2 \cdot 4^m + 1} = \frac{4^{m+1} + 2}{4^{m+1} - 1}.$$

If $z_m = \dfrac{4^{m+1} + 2}{4^{m+1} - 1}$ for some $m \geq 1$, then it follows that $f^m(z_m) = 2$ and hence, $f^n(z_m)$ is
undefined for $n > m$. Therefore, $\lim\limits_{n \to +\infty} f^n(z_m)$ does not exist for these points.

Let

$$S = \{2\} \cup \left\{ \frac{4^{m+1} + 2}{4^{m+1} - 1} : m \in N \right\}.$$

For $z \notin S$, $f^n(z)$ exists for all $n \geq 1$. If $z = 1$, then $z \notin S$ and (3) implies that

$$f^n(1) = \frac{2(4^n - 1) - (2 \cdot 4^n + 1)}{(4^n - 1) - (4^n + 2)}$$

$$= \frac{-3}{-3}$$

$$= 1$$

for all $n \geq 1$. Hence, $\lim\limits_{n \to +\infty} f^n(1) = 1$. For all other values of $z \notin S$,

$$\lim_{n \to +\infty} f^n(z) = \lim_{n \to +\infty} \frac{2(4^n - 1) - (2 \cdot 4^n + 1)z}{(4^n - 1)z - (4^n + 2)}$$

$$= \lim_{n \to +\infty} \frac{2(1 - 4^{-n}) - (2 + 4^{-n})z}{(1 - 4^{-n})z - (1 + 2 \cdot 4^{-n})}$$

$$= \frac{2 - 2z}{z - 1} = -2.$$

Therefore, for $z \notin S$,

$$\lim_{n \to +\infty} f^n(z) = \begin{cases} 1 & \text{if } z = 1 \\ -2 & \text{otherwise} \end{cases}$$

Solution 2 by Henry Ricardo, Westchester Math Circle, NY

We take advantage of the well-known homomorphism between 2×2 matrices and Möbius transformations: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftrightarrow f(z) = \dfrac{az + b}{cz + d}$. In this relation, the n-fold composition $f^n(z)$ corresponds to the nth power of A. Here we are dealing with powers of the matrix $A = \begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix}$.

Now we invoke a known result that is a consequence of the Cayley-Hamilton theorem: If $A \in M_2(C)$ and the eigenvalues λ_1, λ_2 of A are not equal, then for all $n \geq 1$ we have

$$A^n = \lambda_1^n B + \lambda_2^n C, \text{ where } B = \frac{1}{\lambda_1 - \lambda_2}(A - \lambda_2 I_2) \text{ and } C = \frac{1}{\lambda_2 - \lambda_1}(A - \lambda_1 I_2). \, (*)$$

(See, for example, Theorem 2.25(a) in *Essential Linear Algebra with Applications* by T. Andreescu, Birkhäuser, 2014.)

The eigenvalues of the given matrix A are -1 and -4, so we apply $(*)$ to get

$$\begin{aligned} A^n &= \frac{(-1)^n}{3}(A + 4I_2) - \frac{(-4)^n}{3}(A + I_2) \\ &= \left(\frac{(-1)^n - (-4)^n}{3}\right) A + \left(\frac{4 \cdot (-1)^n - (-4)^n}{3}\right) I_2 \\ &= \begin{pmatrix} \frac{1}{3}(-1)^n(1 + 2 \cdot 4^n) & \frac{2}{3}(-1)^n + \frac{2}{3}(-1)^{n+1}4^n \\ \frac{1}{3}(-1)^n + \frac{1}{3}(-1)^{n+1}4^n & \frac{1}{3}(-1)^n(2 + 4^n) \end{pmatrix}. \end{aligned}$$

After some simplification, we see that

$$f^n(z) = \frac{(2 \cdot 4^n + 1)z - 2(4^n - 1)}{(1 - 4^n)z + (4^n + 2)}.$$

Finally, we note that $f^n(1) = 3/3 = 1$; and, for $z \neq 1$, we have

$$\lim_{n \to +\infty} f^n(z) = \lim_{n \to +\infty} \frac{(2 \cdot 4^n + 1)z - 2(4^n - 1)}{(1 - 4^n)z + (4^n + 2)} = \frac{2(z - 1)}{1 - z} = -2.$$

Therefore,

$$\lim_{n \to +\infty} f^n(z) = \begin{cases} 1 & \text{if } z = 1, \\ -2 & \text{if } z \neq 1 \end{cases}.$$

Solution 3 by David E. Manes, Oneonta, NY

We will show by induction that

$$f^{(n)}(z) = \frac{2 - \dfrac{2a_n + 1}{a_n} z}{z - \dfrac{a_n + 1}{a_n}}$$

where $a_n = \dfrac{4^n - 1}{3}$. If $n = 1$, then $a_1 = 1$ and $f^{(1)}(z) = \dfrac{(2 - 3z)}{(z - 2)} = f(z)$. Therefore, the result is true for $n = 1$. Assume the positive integer $n \geq 1$ and the given formula is valid

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for $f^{(n)}(z)$. Then

$$f^{(n+1)}(z) = f(f^{(n)}(z)) = \frac{2 - 3\left(\dfrac{2 - \dfrac{2a_n + 1}{a_n}z}{z - \dfrac{a_n + 1}{a_n}}\right)}{\left(\dfrac{2 - \dfrac{2a_n + 1}{a_n}z}{z - \dfrac{a_n + 1}{a_n}}\right) - 2} = \frac{2z - 2\left(\dfrac{a_n + 1}{a_n}\right) - 6 + 3\left(\dfrac{2a_n + 1}{a_n}\right)z}{2 - \dfrac{2a_n + 1}{a_n}z - 2z + 2\left(\dfrac{a_n + 1}{a_n}\right)}$$

$$= \frac{2a_n z - 2a_n - 2 - 6a_n + 6a_n z + 3z}{2a_n - 2a_n z - z - 2a_n z + 2a_n + 2} = \frac{-2 - 8a_n + (8a_n + 3)z}{-(4a_n + 1)z + (4n + 2)}$$

$$= \frac{2 + 8a_n - (8a_n + 3)z}{(4a_n + 1)z - (4n + 2)} = \frac{2 + 8\left(\dfrac{4^n - 1}{3}\right) - \left(8\left(\dfrac{4^n - 1}{3}\right) + 3\right)z}{\left(4\left(\dfrac{4^n - 1}{3}\right) + 1\right)z - \left(4\left(\dfrac{4^n - 1}{3}\right) + 2\right)}$$

$$= \frac{(-2 + 2 \cdot 4^{n+1}) - (1 + 2 \cdot 4^{n+1})z}{(4^{n+1} - 1)z - (4^{n+1} + 2)}$$

$$= \frac{2 - \left(\dfrac{2 \cdot 4^{n+1} + 1}{4^{n+1} - 1}\right)z}{z - \left(\dfrac{4^{n+1} + 2}{4^{n+1} - 1}\right)} = \frac{2 - \left(\dfrac{\dfrac{2 \cdot 4^{n+1} + 1}{3}}{\dfrac{4^{n+1} - 1}{3}}\right)z}{z - \left(\dfrac{\dfrac{4^{n+1} + 2}{3}}{\dfrac{4^{n+1} - 1}{3}}\right)}$$

$$= \frac{2 - \left(\dfrac{2a_{n+1} + 1}{a_{n+1}}\right)z}{z - \left(\dfrac{a_{n+1} + 1}{a_{n+1}}\right)}$$

where $a_{n+1} = \dfrac{(4^{n+1} - 1)}{3}$. Note that $\dfrac{4^{n+1} + 2}{3} = \dfrac{4^{n+1} - 1}{3} + 1 = a_{n+1} + 1$ and

$$\frac{2 \cdot 4^{n+1} + 1}{3} = \frac{2 \cdot 4^{n+1} - 2}{3} + 1 = 2\left(\frac{4^{n+1} - 1}{3}\right) + 1 = 2a_{n+1} + 1.$$

Hence, the result is true for the integer $n + 1$ so that by the principle of mathematical induction the result is valid for all positive integers n.

For the limit question, note that if $f(z) = z$, then $z = 1$ or $z = -2$. Therefore, one of the fixed points of f is $z = 1$ so that $f^{(n)}(1) = 1$ for each positive integer n and $\lim_{n \to +\infty} f^{(n)}(1) = 1$. Moreover, observe that

$$\lim_{n \to +\infty} \frac{1}{a_n} = \lim_{n \to +\infty} \frac{3}{4^n - 1} = 0.$$

Therefore, if $z \neq 1$, then

$$\lim_{n\to+\infty} f^{(n)}(z) = \lim_{n\to+\infty} \left(\frac{2 - \dfrac{2a_n + 1}{a_n}z}{z - \dfrac{a_n + 1}{a_n}}\right) = \frac{\left(2 - \lim_{n\to+\infty}\left(2 + \dfrac{1}{a_n}\right)z\right)}{\left(z - \lim_{n\to+\infty}\left(1 + \dfrac{1}{a_n}\right)\right)} = \frac{2 - 2z}{z - 1} = -2.$$

Hence,

$$\lim_{n\to+\infty} f^{(n)}(z) = \begin{cases} 1, & \text{if } z = 1, \\ -2, & \text{if } z \neq 1. \end{cases}$$

Solution 4 by Jeremiah Bartz, University of North Dakota, Grand Forks, ND

Recall the map $f(z) = \dfrac{az + b}{cz + d} \mapsto \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ gives a group isomorphism between group of fractional linear transformations

$$\left\{ f : f(z) = \frac{az + b}{cz + d} \text{ where } a, b, c, d \in C \text{ and } ad - bc \neq 0 \right\}$$

under function composition and the group

$$GL(2, C) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in C \text{ and } ad - bc \neq 0 \right\}$$

under matrix multiplication.

To compute $f^n(z)$, let $M = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}$. Using induction, we show

$$M^n = \frac{(-1)^n}{3} \begin{bmatrix} 2^{2n+1} + 1 & -2^{2n+1} + 2 \\ -4^n + 1 & 4^n + 2 \end{bmatrix}.$$

Observe $M^1 = \dfrac{-1}{3} \begin{bmatrix} 2^3 + 1 & -2^3 + 2 \\ -3 & 6 \end{bmatrix} = \dfrac{-1}{3} \begin{bmatrix} 9 & -6 \\ -3 & 6 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix}.$

Assume

$$M^n = \frac{(-1)^n}{3} \begin{bmatrix} 2^{2n+1} + 1 & -2^{2n+1} + 2 \\ -4^n + 1 & 4^n + 2 \end{bmatrix}$$

and observe

$$\begin{aligned} M^{n+1} &= M^n M \\ &= \frac{(-1)^n}{3} \begin{bmatrix} 2^{2n+1} + 1 & -2^{2n+1} + 2 \\ -4^n + 1 & 4^n + 2 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} \\ &= \frac{(-1)^n}{3} \begin{bmatrix} -3(2^{2n+1} + 1) + (-2^{2n+1} + 2) & 2(2^{2n+1} + 1) - 2(-2^{2n+1} + 2) \\ -3(-4^n + 1) + (4^n + 2) & 2(-4^n + 1) - 2(4^n + 2) \end{bmatrix} \\ &= \frac{(-1)^{n+1}}{3} \begin{bmatrix} 2^{2(n+1)+1} + 1 & -2^{2(n+1)+1} + 2 \\ -4^{n+1} + 1 & 4^{n+1} + 2 \end{bmatrix}. \end{aligned}$$

Using the aforementioned group isomorphism and simplifying, we conclude

$$f^n(z) = \frac{(2^{2n+1} + 1)z - 2^{2n+1} + 2}{(-4^n + 1)z + 4^n + 2} = \frac{(2 \cdot 4^n + 1)z + (2 - 2 \cdot 4^n)}{(1 - 4^n)z + (2 + 4^n)}.$$

Notice that the map $f^n(z)$ is undefined for $z = \dfrac{4^k + 2}{4^k - 1}$ where $1 \le k \le n$. Consequently $\lim\limits_{n \to +\infty} f(z)$ does not exist for these values of z. Furthermore,

$$
\begin{aligned}
\lim_{n \to +\infty} f^n(z) &= \lim_{n \to +\infty} \frac{(2 \cdot 4^n + 1)z + (2 - 2 \cdot 4^n)}{(1 - 4^n)z + (2 + 4^n)} \\
&= \lim_{n \to +\infty} \frac{\left(2 + \frac{1}{4^n}\right)z + \left(\frac{2}{4^n} - 2\right)}{\left(\frac{1}{4^n} - 1\right)z + \left(\frac{2}{4^n} + 1\right)} \\
&= \frac{2z - 2}{-z + 1} \\
&= -2\left(\frac{1-z}{1-z}\right).
\end{aligned}
$$

Note $f(1) = 1$ so $f^n(1) = 1$ for all $n \ge 1$. It follows that

$$
\lim_{n \to +\infty} f(z) = \begin{cases} \text{DNE} & \text{if } z = \dfrac{4^n + 2}{4^n - 1} \text{ where } n \in Z_{>0} \\ 1 & \text{if } z = 1 \\ -2 & \text{otherwise.} \end{cases}
$$

$$
(\text{DNE} \;=\; \text{does not exist})
$$

Comment by Editor : **David Stone and John Hawkins of Georgia Southern University** stated the following in their solution: "The appearance of so many sums of powers of 4 prompts us to offer a candidate for the cutest representation of $f^{(n)}(z)$:

$$
f^{(n)}(z) = \frac{(2 \cdot 111\ldots\ldots 1_4 + 1)\, z - 2 \cdot 111\ldots 1_4}{-111\ldots\ldots 1_4 z + (111\ldots 1_4 + 1)},
$$

where each of the base 4 repunits has $n - 1$ digits."

Solution 5 by Toshihiro Shimizu, Kawasaki, Japan

Let $f^n(z) = \dfrac{a_n z + b_n}{c_n z + d_n}$. Then, we have

$$
\begin{aligned}
\frac{a_{n+1} z + b_{n+1}}{c_{n+1} z + d_{n+1}} &= f^{n+1}(z) \\
&= f^n\left(\frac{2 - 3z}{z - 2}\right) \\
&= \frac{(b_n - 3a_n)\, z + 2\,(a_n - b_n)}{(d_n - 3c_n)\, z + 2\,(c_n - d_n)}
\end{aligned}
$$

Therefore, we have $a_{n+1} = b_n - 3a_n$, $b_{n+1} = 2a_n - 2b_n$ and $c_{n+1} = d_n - 3c_n$, $d_{n+1} = 2c_n - 2d_n$. Since $f^0(z) = z$, $a_0 = 1, b_0 = c_0 = 0$ and $d_0 = 1$. Since $b_n = a_{n+1} + 3a_n$, we have

$$
\begin{aligned}
a_{n+2} + 3a_{n+1} &= 2a_n - 2\,(a_{n+1} + 3a_n) \\
a_{n+2} + 5a_{n+1} + 4a_n &= 0
\end{aligned}
$$

and $a_1 = b_0 - 3a_0 = -3$. Thus, we have

$$a_n = \frac{1}{3}(-1)^n + \frac{2}{3}(-4)^n$$

$$b_n = a_{n+1} + 3a_n$$

$$= \frac{1}{3}(-1)^{n+1} + \frac{2}{3}(-4)^{n+1} + (-1)^n + 2(-4)^n$$

$$= \frac{2}{3}(-1)^n - \frac{2}{3}(-4)^n.$$

Similarly, we have $c_{n+2} + 5c_{n+1} + 4c_n = 0$ and $c_1 = d_0 - 3c_0 = 1$. Thus, we have

$$c_n = \frac{1}{3}(-1)^n - \frac{1}{3}(-4)^n$$

$$d_n = c_{n+1} + 3c_n$$

$$= \frac{2}{3}(-1)^n + \frac{1}{3}(-4)^n$$

Therefore,

$$f^n(z) = \frac{((-1)^n + 2(-4)^n)z + (2(-1)^n - 2(-4)^n)}{((-1)^n - (-4)^n)z + (2(-1)^n + (-4)^n)}.$$

If $z \neq 1$, we have

$$f^n(z) = \frac{\left(\left(\frac{1}{4}\right)^n + 2\right)z + \left(2\left(\frac{1}{4}\right)^n - 2\right)}{\left(\left(\frac{1}{4}\right)^n - 1\right)z + \left(2\left(\frac{1}{4}\right)^n + 1\right)}$$

$$\to \frac{2z - 2}{-z + 1}$$

$$= -2 \quad (n \to +\infty).$$

If $z = 1$, the value of $f^n(z)$ is always 1 and its limit is also 1.

Solution 6 by Kee-Wai Lau, Hong Kong, China

It can easily be proved by induction that

$$f^n(z) = \frac{2(2^{2n} - 1) - (2^{2n+1} + 1)z}{(2^{2n} - 1)z - 2(2^{2n-1} + 1)},$$

whenever $z \notin S_n$, where $S_n = \{2\} \cup \left\{ \frac{2(2^{2k-1} + 1)}{2^{2k} - 1} : k = 1, 2, 3, \cdots, n \right\}$.

Clearly, $\lim\limits_{n \to \infty} f^n(1) = 1$ and if $z \notin \mathbf{T}$, where $\mathbf{T} = \{1, 2\} \cup \left\{ \frac{2(2^{2k-1} + 1)}{2^{2k} - 1}, k = 1, 2, 3 \cdots \right\}$, then $\lim\limits_{n \to \infty} f^n(z) = -2$.

Also solved by Arkady Alt, San Jose, CA; Hatef I. Arshagi, Guilford Technical Community College, Jamestown, NC; Brian D. Beasley, Presbyterian College, Clinton, SC; Brian Bradie, Christopher Newport University, Newport News,VA; Bruno Salgueiro Fanego Viveiro, Spain; Ed Gray, Highland Beach, FL; Moti Levy (two solutions), Rehovot, Israel; Francisco Perdomo and Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain; Trey Smith, Angelo State University, San Angelo, TX; Albert Stadler, Herrliberg, Switzerland; David Stone and John Hawkins, Georgia Southern University, Statesboro, GA, and the proposer.